Notes on Semisimple Algebras, Jacobson Radical, and Group Algebras

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Contents

1 Introduction

These notes survey several foundational topics in the theory of rings and modules, focusing on:

- The notion of semisimple (a.k.a. Jacobson semisimple) rings and modules.
- The Jacobson radical and its properties.
- Group algebras $F[G]$ and Maschke's Theorem.
- The center of a ring and how the radical intersects it.
- Nakayama's Lemma.
- Some brief references to field extensions, Artin–Schreier polynomials, and Steinitz's work on field theory.

We assume familiarity with basic ring theory (ideals, modules, etc.) but will keep the exposition accessible to upper-level undergraduates.

2 Preliminaries and Definitions

2.1 Jacobson Radical

Definition 2.1 (Jacobson Radical)**.** *Let A be a ring (associative with unity). The* Jacobson radical *of A, denoted $J(A)$, is defined as*

$$J(A) \;=\; \bigcap \{\, I \subseteq A : I \text{ is a maximal left ideal of } A\}.$$

Equivalently (by a standard theorem), $J(A)$ is the intersection of the annihilators of all simple A-modules:

$$J(A) \;=\; \bigcap_{V \text{ simple } A\text{-mod}} \operatorname{Ann}(V).$$

Remark 2.2. *If A is finite-dimensional over a field F, then $J(A)$ is a nilpotent ideal, meaning $(J(A))^n = 0$ for some $n \geq 1$. In fact, for finite-dimensional algebras over a field, $J(A)$ coincides with the* radical *in the sense of classical Artin–Wedderburn theory.*

2.2 Semisimple Rings and Modules

Definition 2.3 (Semisimple (Artin–Wedderburn) Ring)**.** *A ring A is called* semisimple *(or* Jacobson semisimple*) if its Jacobson radical $J(A)$ is zero. Equivalently (if A is Artinian), A is semisimple if it is isomorphic to a finite direct product of matrix rings over division rings.*

Definition 2.4 (Completely Reducible / Semisimple Module)**.** *An A-module V is said to be* completely reducible *(or* semisimple*) if it is a direct sum of simple A-modules.*

Theorem 2.5 (Artin–Wedderburn)**.** *Let A be a semisimple ring that is finite-dimensional over a field F. Then A is isomorphic to a finite direct product of matrix algebras over division rings. Concretely,*

$$A \;\cong\; M_{n_1}(D_1) \times M_{n_2}(D_2) \times \cdots \times M_{n_k}(D_k),$$

where each D_i is a (possibly) different division algebra over F.

3 Group Algebras and Maschke's Theorem

Let F be a field, and G a finite group. The *group algebra* $F[G]$ is the F-vector space with basis $\{g : g \in G\}$ and multiplication induced by the group operation extended F-linearly:

$$\Big(\sum_{g\in G} \alpha_g \, g\Big) \Big(\sum_{h\in G} \beta_h \, h\Big) \;=\; \sum_{g,h\in G} \alpha_g \, \beta_h \, (gh).$$

3.1 Maschke's Theorem

Theorem 3.1 (Maschke)**.** *Suppose G is a finite group and F is a field whose characteristic does* not *divide $|G|$. Then the group algebra $F[G]$ is semisimple as a ring. In other words, $J(F[G]) = 0$.*

Remark 3.2. *The key reason is that $|G|$ is invertible in F, so the element*

$$e = \frac{1}{|G|} \sum_{g\in G} g$$

is a central idempotent in $F[G]$, and this leads to a decomposition argument showing every $F[G]$-module is semisimple.

Corollary 3.3. *If* $\operatorname{char}(F) = 0$*, then automatically* $F[G]$ *is semisimple. For example, over* $\mathbb{C}$ *or* $\mathbb{Q}$*, all finite group algebras are semisimple.*

3.2 Case $\operatorname{char}(F)$ Divides $|G|$

If $\operatorname{char}(F)$ divides $|G|$, then $|G|$ is 0 in F, so $1/|G|$ does not exist in F. In this situation, $F[G]$ is *not* semisimple. One can show:

$$\sum_{g \in G} g \ \in \ J(F[G]),$$

and indeed $J(F[G]) \neq 0$. Often one proves that $\left(\sum_{g \in G} g\right)$ is nilpotent or that it generates a nonzero nilpotent ideal, depending on further structure.

4 Nilpotent Ideals and Containment in the Radical

Definition 4.1 (Nilpotent Ideal)**.** *An ideal* I *in a ring* A *is* nilpotent *if there exists* $n \geq 1$ *such that* $I^n = 0$*.*

Proposition 4.2. *If* I *is a nilpotent ideal of* A*, then* $I \subseteq J(A)$*.*

Sketch of Proof. Let $I \subseteq A$ be nilpotent. We want to show I annihilates every simple A-module V. Indeed, if $I^n = 0$ for some n, then for any $v \in V$, the submodule $I^n V = 0$, so eventually one can see I acts trivially on V (by a standard descending argument on powers of I or by considering the fact that I must lie in every maximal left ideal). Hence $I \subseteq \operatorname{Ann}(V)$ for every simple V. Thus

$$I \ \subseteq \ \bigcap_{V \text{ simple}} \operatorname{Ann}(V) \ = \ J(A).$$

□

5 Center of a Ring and Intersection with the Radical

Definition 5.1 (Center of a Ring)**.** *The* center *of a ring* A *is*

$$Z(A) \ = \ \{\, z \in A : za = az \text{ for all } a \in A\}.$$

It is straightforward to see that $Z(A)$ is a commutative subring of A. Moreover, $Z(A)$ is an ideal in A if and only if $Z(A) = A$ itself (i.e. A is commutative).

Proposition 5.2 (Intersection of Jacobson Radical with Center)**.** *We always have*

$$J(A) \cap Z(A) \ \subseteq \ J(Z(A)).$$

Furthermore, for finite-dimensional A*, the intersection* $J(A) \cap Z(A)$ *is a nilpotent ideal in* $Z(A)$*, hence contained in the nilradical of* $Z(A)$*.*

Idea of Proof. By definition, $J(A) \cap Z(A)$ consists of central elements of A which are in the radical. Such elements are also in the radical of the subring $Z(A)$, denoted $J(Z(A))$. In a finite-dimensional setting, this subring is Artinian, and so $J(Z(A))$ is nilpotent. Hence every element in $J(A) \cap Z(A)$ is nilpotent (in fact, that entire intersection forms a nilpotent ideal in $Z(A)$). □

6 Nakayama's Lemma

Definition 6.1 (Nakayama's Lemma (Classical Form))**.** *Let A be a ring with Jacobson radical $J(A)$, and let M be a finitely generated left A-module. Then*

$$J(A)\,M \;=\; M \quad\Longrightarrow\quad M = 0.$$

Equivalently, if N is a submodule of M such that $M = N + J(A)M$, then $M = N$.

Theorem 6.2 (Nakayama's Lemma)**.** *If A is a ring with Jacobson radical $J(A)$ and M is a finitely generated A-module such that $M = J(A)M$, then $M = 0$. In a more general form, for any submodule $N \subseteq M$, if $M = N + J(A)M$, then $M = N$.*

Sketch of Proof. The standard proof involves picking a finite set of generators for M and using the fact that $1 - x$ is invertible if $x \in J(A)$. See any standard textbook in ring theory (e.g. *Atiyah–Macdonald*, *Lang's Algebra*) for details. □

7 Field-Theoretic Asides: Artin–Schreier Polynomials and Steinitz

Remark 7.1. *In the handwritten notes, there were references to towers of field extensions (Steinitz), Artin–Schreier polynomials of the form $x^p - x - \alpha$ over fields of characteristic p, and so on. These are classical topics in field theory, showing how certain extensions are constructed in characteristic p.*

- **Steinitz's Theorem on Field Extensions (1910):** Ernst Steinitz laid the foundation of modern field theory, introducing the notions of prime fields, algebraic closure, separability, etc.

- **Artin–Schreier Extensions:** In characteristic $p > 0$, an extension $F(\alpha)/F$ is called Artin–Schreier if α satisfies a polynomial $x^p - x - c$ for some $c \in F$. These are analogous to cyclotomic extensions in characteristic 0.

8 Summary of Key Points

- **Jacobson Radical:** $J(A)$ is the intersection of all maximal left ideals, and for finite-dimensional algebras, it is nilpotent. Nilpotent ideals lie in $J(A)$.

- **Semisimple Algebras:** A is semisimple (a.k.a. has zero Jacobson radical) precisely when every A-module is semisimple (completely reducible). By Artin–Wedderburn, finite-dimensional semisimple algebras over a field decompose as products of matrix algebras over division rings.

- **Group Algebras:** For a finite group G over a field F, $F[G]$ is semisimple if and only if $\mathrm{char}(F)$ does not divide $|G|$ (Maschke's Theorem).

- **Nakayama's Lemma:** A fundamental result on modules over rings with nonzero Jacobson radical, ensuring that $J(A)M = M$ forces M to be zero (when M is finitely generated).

- **Center and Intersection:** $J(A) \cap Z(A)$ is contained in $J(Z(A))$, and in finite-dimensional settings, that intersection is nilpotent in the center.

References for Further Reading

- T. Y. Lam, *A First Course in Noncommutative Rings*, Graduate Texts in Mathematics.
- S. Lang, *Algebra*, Graduate Texts in Mathematics.
- M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*.
- N. Jacobson, *Structure of Rings*.
- E. Artin, *Geometric Algebra*, for a historical perspective on Artin–Schreier.
- E. Steinitz, *Algebraische Theorie der Körper*, 1910 (classical paper on field theory).